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THE GLOBAL FAILURE OF LOCAL PREFERENCE OPTIMIZATION

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Abstract—We consider the problem of a consumer with smooth preferences who does not *a priori* know what his optimal feasible consumption bundle is, but attempts to find it by continuously moving in some direction of increasing preferences, starting with an arbitrary initial bundle. Previous analyses based on local arguments have shown that, under plausible departures from the textbook ideal, such a consumer may get locked into a stable limit cycle and thus fail to reach his consumption optimum. In this article, we give a first look at this problem from the global viewpoint of the Poincaré-Bendixson Theorem. Numerical analyses suggest that global failures of the hypothesized consumer local search process may be fairly common and robust occurrences.

1. INTRODUCTION

In this article, we consider a consumer which departs from the textbook ideal in the following three aspects:

(a) The consumer's preferences are intransitive. Thus, they do not admit a utility representation (they are not integrable).
(b) The consumer does not *a priori* know what his optimal feasible consumption bundle is, but attempts to find it by continuously moving in a direction of locally increasing preferences, starting with an arbitrary initial bundle.
(c) In his search for the optimum, the consumer is possibly inexact in the sense that he does not necessarily follow the direction of *fastest* preferences increases $\mathbf{g}$ (the utility gradient in the integrable case), but may follow any direction he associates with a utility increase (that is, in our terminology, any *positive direction* $\mathbf{m}$).

The consumer's preferences are assumed to belong to the very general class of *Non-Transitive Smooth Preferences*, NTSP, developed by Al-Najjar in [1]. Save for their including the possibility of intransitivity, NTSP are perfectly standard and possess all the "nice" properties usually required from preferences. They arguably represent the most general class of well-behaved preferences yet introduced in the literature and include the most important standard preferences as special cases.

In a previous article [2], we used the (local) Hopf bifurcation theorem to show that in a three-goods NTSP world, following a positive direction may result in the consumer getting locked into a stable limit cycle, thus failing to reach his optimum. An attempt at generalizing the results presented in [2] to the four goods case was carried out in [3], still from the local perspective of Hopf's bifurcation theorem. By contrast, the present article represents a first attempt to attack the problem from the Poincaré-Bendixson Theorem viewpoint, to obtain a global perspective on

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the occurence of consumer failures in a three-goods world, that is, when the consumer's search is carried out in the plan. It may be worth noting that the Poincaré-Bendixson theorem, and thus the presented results, do not have any direct generalization to higher-dimensional spaces. Cycles cannot occur in an environment with less than three goods.

The article is organised as follows. We first describe succintly the framework in which the analysis is conducted and attempt to justify the underlying hypotheses, closely following [2,3] to which the reader is referred for more detail (Section 2). We then set the stage for an analysis using the Poincaré-Bendixson Theorem and present the analytical and numerical results obtained thus far (Section 3). The paper ends with some cursory conclusions (Section 4).

2. PREFERENCES AND OPTIMIZATION

2.1. Consumers' Preferences and Choice

The basic framework of this article is completely standard. We consider a consumer who aims at chosing the "best possible" ***consumption bundle*** $\mathbf{x} \in X = \mathbb{R}^{n+1}_{++}$ available to him in the ***commodity space*** X given his (fixed) ***income*** $\beta \in \mathbb{R}_{++}$ and the prevailing ***price vector*** $\mathbf{p} \in \mathbb{R}^{n+1}_{++}$. The coordinate x_i of $\mathbf{x}$, $i = 1, \ldots, n+1$, is the quantity of good i that the consumer will buy, that is, consume if he choses the bundle $\mathbf{x}$.

Here, and in the following, upper case letters are used to designate sets and matrices; lower case letters are reserved for vectors and for elements of matrices and vectors. We use $\mathbf{x}\,\mathbf{y}$ to represent the scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$. Scalars are denoted by Greek letters. The subscript $+$ indicates weak, and $++$ strict positivity. That is, $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \neq 0,\ x_i \geq 0,\ i = 1, \ldots, n\}$, and $\mathbb{R}^{n+1}_{++} = \{\mathbf{x} \in \mathbb{R}^n \mid x_i > 0,\ i = 1, \ldots, n\}$. Thus, the assumption $X = \mathbb{R}^{n+1}_{++}$ implies that the consumer will always buy a positive quantity (possibly very small) of every good. This is a largely innocuous assumption frequently made in economics, mostly for technical reasons. We use it in this paper since it underlies our reference framework, the NTSP. However, our argumentation would remain valid under the hypothesis $X = \mathbb{R}^{n+1}_{+}$.

As usual in the study of consumers' choice, we suppose that the consumer can rank any pair of bundles $\mathbf{x}$ and $\mathbf{y} \in X$ by determining whether or not it is true that "$\mathbf{x}$ is at least as good as $\mathbf{y}$," that is, is *weakly preferred* to $\mathbf{y}$. Formally, we assume that the consumer has a ***preference relation***, that is, a binary relation $P \subset \mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++}$ with the interpretation:

– For all $\mathbf{x},\ \mathbf{y} \in X$, $\mathbf{x}\ P\ \mathbf{y}$ means that $\mathbf{x}$ is *weakly preferred* to $\mathbf{y}$.

The preference P is assumed to be total and reflexive, and to satisfy ***desirability***, that is, $\mathbf{x} > \mathbf{y} \Rightarrow \mathbf{x}\ P\ \mathbf{y}$. (Here and in the following, $\mathbf{x} > \mathbf{y}$ means: $x_i > y_i$, for all i.) Furthermore, it is assumed to define on any ***budget set*** $B = B(\mathbf{p}, \beta) \equiv \{\mathbf{x} \in X \mid \mathbf{p}\,\mathbf{x} \leq \beta\}$ (that is, on the set of all consumption bundles that, at the current prices $\mathbf{p}$, do not cost more than the consumer's income β) a unique ***consumption optimum*** $\mathbf{x}^* = \mathbf{x}^*(B)$ such that:

$$\mathbf{x}* \in B(\mathbf{p}, \beta) \text{ and, for all } \mathbf{x} \in B(\mathbf{p}, \beta),\ \mathbf{x}^*\ P\ \mathbf{x}. \tag{1}$$

Given his current budget set B, a rational consumer should actually choose the consumption optimum $\mathbf{x}^* = \mathbf{x}^*(B)$. The consumer's vector-valued ***demand correspondence*** h, defined by $\mathbf{x}^* \in h(\mathbf{p}, \beta)$ IFF $\mathbf{x}^*$ satisfies (1), describes the relationship between income and prices on the one hand, the consumption optimum on the other. Due to desirability, $\mathbf{x}^*$ will lie in the consumer's ***budget constraint*** $Y = Y(\mathbf{p}, \beta) \equiv \{\mathbf{x} \in X \mid \mathbf{p}\mathbf{x} = \beta\}$, that is, in the set of all consumption bundles that exactly exhaust the consumer's income β at the given commodity prices $\mathbf{p}$. In other words, desirability implies that the consumer would always prefer to consume more of any good, and thus at the optimum will not save any part of his income. For that reason, see also [2,3], we will restrict our analysis to the possible consumer search behavior within Y (and not, within X). Note that Y is an open and convex subset of a (translation of) the n-dimensional hyperplane in X with normal $\mathbf{p}$, and is thus of dimension n.

2.2. Transitivity

Textbook economic theory makes the assumption that P is transitive, that is:

$$\mathbf{x}\ P\ \mathbf{y}\ \&\ \mathbf{y}\ P\ \mathbf{z} \Rightarrow \mathbf{x}\ P\ \mathbf{z}. \tag{2}$$

Indeed, many decision-scientists (and possibly most economists) do consider (2) as a fundamental principle of normative decision theory. Nonetheless, the hypothesis of transitive preferences has been abundantly criticized in recent years, for being both empirically counterfactual and theoretically unnecessary; see, among others, the most recent and authoritative review article [4]. Thus, at the empirical level, the work reported among others in [5–7] suggests that transitivity is unlikely to be observed in complex decision-making environments. While unproblematic in a choice situation with few, simple alternatives, it appears too stringent a requirement to be followed by a human decision-maker in a situation involving numerous, multi-dimensional options. Furthermore, at the theoretical level, Sonnenschein [8] showed as early as in 1971 that the transitivity axiom can be dispensed with in proving many important results of the theory of competitive equilibrium. Other classical theoretical results include [9–11]. Also, diverse non-transitive extensions of standard preferences have been proposed since the early 1970's, culminating in Al-Najjar's Non-Transitive Smooth Preferences, NTSP.

Al-Najjar's work gives an additional strong argument for concentrating on the non-transitive and, more specifically, on the NTSP case. In [1, Proposition 4.1], it is shown that NTSP, while satisfying (with the exception of transitivity) all the demands usually placed on well-behaved preferences, include as a special case the "standard" transitive smooth preferences *à la* Debreu [12]—the mainstay of modern consumer analysis. Furthermore, every preference in NTSP generates a demand function which is extremely well behaved. (In particular, it is smooth and satisfies the weak axiom of revealed preferences.) Finally, the demand functions generated by the standard preferences constitute a closed and nowhere dense subset of the set of demand functions generated by preferences in NTSP. By contrast, small perturbations to a demand function generated by a preference in NTSP yield a demand function which is also generated by a preference in NTSP. That is, NTSP are robust with regard to small perturbations, while standard transitive smooth preferences are not. In other words, transitive preferences and the corresponding demand functions appear as isolated singularities in a generically non-transitive world. Therefore, it seems unlikely that a demand behavior corresponding exactly to a transitive preference may ever be observed in practice—some degree of intransitivity will presumably always be present.

2.3. Non-Transitive Smooth Preferences NTSP

Keeping in mind that preferences in NTSP are standard (that is above all, complete, reflexive binary orderings on $\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++}$ satisfying desirability), the NTSP's property crucial for us is the following. If a preference P is a typical element of NTSP, then it satisfies [1, Chapter 2]:

STRONG CONVEXITY (SC). *For all* $\mathbf{x} \in X$ *there exists (up to a scalar multiple) a unique vector* $\mathbf{g} = \mathbf{g}(\mathbf{x}) \in \mathbb{R}^{n+1}_{++}$ *such that:*

$$[\mathbf{x}\ P\ \mathbf{y}\ \&\ \mathbf{x} \neq \mathbf{y} \Rightarrow \mathbf{g}(\mathbf{x})\,\mathbf{x} - \mathbf{g}(\mathbf{x})\,\mathbf{y} > 0].$$

To understand the substantive meaning of (SC), first recall that the preferences are "primitives" in consumption theory. However, the standard theory of smooth preferences [12] is typically developed in terms of utility functions, that is, of real-valued functions $\omega : X \to \mathbb{R}$ "representing" the preferences P in the sense that:

$$\forall \mathbf{x}, \mathbf{y} \in X,\ \mathbf{x}\ P\ \mathbf{y} \iff \omega(\mathbf{x}) \geq \omega(\mathbf{y}). \tag{3}$$

The contours of a utility function thus correspond to the set of all bundles $\mathbf{x}$ which are, from the consumer's viewpoint, indifferent to a given bundle, that is, to ***indifference sets***. Under the usual assumptions, the indifference sets are "thin," continuous, stricly convex indifference curves or surfaces. The unique consumption optimum $\mathbf{x}^*(Y)$ is then the unique point where Y is tangent to an indifference surface, and the demand correspondence h is a smooth function.

However, not all preferences P have a utility representation ω, that is, are *integrable*. This is the case, in particular, with intransitive preferences. Accordingly, in a non-transitive world, indifference curves do not exist. The assumption (SC) yet insures that in any point $\mathbf{x}$ in X the weakly preferred set $P(\mathbf{x}) \equiv \{\mathbf{y} \in X \mid \mathbf{y}\ P\ \mathbf{x}\}$ has a unique supporting hyperplane. Given preferability, there is then (up to a scalar multiple) a unique positive vector $\mathbf{g}(\mathbf{x}) \in \mathbb{R}^n_{++}$ orthogonal to the supporting hyperplane, which can be interpreted as the consumer's "preferred direction." Indeed, if P can be represented by a strictly quasi-concave and differentiable utility function ω which has no critical points, then P will satisfy (SC) and $\mathbf{g}(\mathbf{x})$ will be the gradient of ω at $\mathbf{x}$. In the strictly non-transitive case, however, it is not possible to represent P by a utility function and $\mathbf{g}$ is not, *stricto sensu*, a gradient. For simplicity, we will nonetheless continue to call it so.

Condition (SC) thus insures the existence of a well-defined "gradient" vector field $\mathbf{g}(\mathbf{x})$. Under the diverse technical assumptions underlying NTSP, $\mathbf{g}$ is continuous [1, Lemma 2.1 (viii), p. 20], and defines a well-behaved, smooth demand function h.

The gradient field $\mathbf{g}(\mathbf{x})$, however, is of no direct interest for us since, as previously mentioned, we only consider search processes within Y. We are, therefore, naturally brought to concentrate on the (orthogonal) projection $\mathbf{q} = \mathbf{q}(\mathbf{x})$ of $\mathbf{g}(\mathbf{x})$, $\mathbf{x} \in Y$, on Y along the given price vector $\mathbf{p}$. One can show [1, Section 2.3] that a vector field $\mathbf{q}(\mathbf{x})$ representing the projection on Y of the "gradient" field $\mathbf{g}(\mathbf{x})$, $\mathbf{x} \in Y$, of an NTSP will satisfy:

A1. *The vector field* $\mathbf{q}(\mathbf{x})$ *is* C^1 *and has a unique fixed point* $\mathbf{x}^* = \mathbf{0}$ *over* Y;

A2. *Its Jacobian* $D\mathbf{q}(\mathbf{x})$ *is quasi negative-definite, that is,* $\mathbf{v}D\mathbf{q}(\mathbf{x})\mathbf{v} < 0$ *for all* $\mathbf{x} \in Y$ *and* $\mathbf{v} \in Y$, *with* $\mathbf{v} \neq \mathbf{x}^*$ *and* $\mathbf{v}\,\mathbf{g}(\mathbf{x}^*) = 0$.

Furthermore we have:

B. *The Jacobian* $D\mathbf{q}(\mathbf{x})$ *is symmetric for all possible budget constraints* Y *(and the preferences have a utility representation, i.e., are* ***integrable****) IFF the preferences are transitive.*

As previously noted, NTSP includes both transitive and non-transitive preferences, intransitivity being given whenever the Jacobian $D\mathbf{q}(\mathbf{x})$, $\mathbf{x} \in Y$, is non-symmetric for at least one budget constraint Y within X. We shall call in the following *non-integrable* (*integrable*) preferences, those intransitive preferences in NTSP which have a non-symmetric (symmetric) Jacobian on the "current" budget constraint Y to which our analysis is restricted.

One finds back here, albeit in a new form, familiar results from consumer theory. To replace (A1), for example, in a more familiar framework, note that $D\mathbf{q}(\mathbf{x})$ is the utility's Hessian on Y in the special case where $\mathbf{g}$ is the gradient of some C^2 utility function.

In the rest of the paper, the notation $\mathbf{A}$ is used to designate the Jacobian $D\mathbf{q}(\mathbf{x})$, evaluated at $\mathbf{x}^*$, and the notation $\mathbf{y}$ for $\mathbf{x} - \mathbf{x}^*$, $\mathbf{x} \in Y$, with $\mathbf{y}^* = \mathbf{0}$.

We now turn to modelling the consumer's search for the optimum.

2.4. Positive Local Search Processes

Consider a consumer faced with a choice situation new or unfamiliar to him, in the sense (among others) that the current budget set differs considerably from the old one, due to changes in prices, in income, and/or in the list of available goods. In all likelihood, the new consumption optimum will also strongly differ from the past one. If the new budget set includes more than a few straightforward alternatives, one cannot expect the consumer to be from the onset able to globally rank all his consumption possibilities and thus recognize immediately and effortlessly his new optimum. More realistically, one may suppose that a consumer will typically use some "reasonable" search procedure to find his global optimum within the given budget set—a point recognized by a number of economists, see among others [13], but nonetheless ignored in most textbook presentations of consumer theory. In this paper, we assume that the consumer uses exclusively local information to guide him in his search efforts. Specifically, we hypothesize the following search process: at each $\mathbf{y}$ in Y, the consumer is able to recognize if he prefers, does not prefer, or is indifferent to, moving some small positive distance from $\mathbf{y}$ in any direction

$\overrightarrow{\mathbf{y}\mathbf{y}'}$, $\mathbf{y}' \in Y$. However, he may not not necessarily be able to recognize the exact preferred direction at $\mathbf{y}$, that is, $\mathbf{q}(\mathbf{y})$.

Given this limited informational capability and precision, the best that a consumer will be able to do is to start a search process at an arbitrary bundle $\mathbf{y}^0 \neq \mathbf{y}^*$ in Y, and from there on make continuous infinitesimal utility-increasing adjustments until the global optimum is reached.

Formally, we shall postulate that the consumer search obeys a *(strictly) positive search process* starting at $\mathbf{y}^0$: Let $\mathbf{y} : [0,\infty) \to Y$ be any continuous function that is also C^1 on $(0,\infty)$. Such a $\mathbf{y}$ will be called a *path*. A path is the outcome of a strictly positive process starting at $\mathbf{y}^0$ if:

$$\frac{d\mathbf{y}_t}{dt}\,\mathbf{q}(\mathbf{y}_t) > 0 \quad \text{for all} \quad \mathbf{y}_t \neq \mathbf{y}^*; \qquad \mathbf{y}_0 = \mathbf{y}^0. \tag{4}$$

A time derivative $\frac{d\mathbf{y}_t}{dt}$ satisfying (4) will be called a *positive direction* (with respect to $\mathbf{q}(\mathbf{y}_t)$ resp. $\mathbf{y}_t$). Intuitively, a positive direction $\frac{d\mathbf{y}_t}{dt}$ is a "desirable" direction from $\mathbf{y}_t$, see [13, pp. 203–205; 14, p. 552; 15, p. 118], quoted in [16, p. 11]. In that sense, (4) defines a "reasonable" search procedure in the budget set based on local information.

This completes the definition of the framework in which our analysis will be conducted. One may note that this framework is purely deterministic. It hypothesizes that the consumer has no global perception of his preferences on the budget constraint, and a somewhat imperfect local one. Moreover, the consumer does not learn. Thus, he will be unable to recognize being attracted towards a limit cycle, should this be the case.

3. CONSUMER CYCLES AND THE POINCARÉ-BENDIXSON THEOREM

3.1. The General Background

The restrictions imposed on the field $\mathbf{q}$ by economic theory in general, and NTSP theory in particular, are extremly weak. In the three-goods case, all we know from (A–B) is that a C^1 field:

$$\mathbf{q}(\mathbf{y}) = \mathbf{A}\,\mathbf{y} + O(|\mathbf{y}|^2), \qquad \mathbf{y} \in Y \tag{5}$$

is a candidate for the projected gradient field of a NTSP if it is C^1 and has a unique equilibrium at the origin, and if the coefficients of the matrix $\mathbf{A}$ of its *linear part* $\mathbf{A}\,\mathbf{y}$ satisfy:

$$a_{11} < 0, \qquad a_{22} < 0, \tag{6a}$$

$$a_{11}\,a_{22} - \left[\frac{1}{2}(a_{12} + a_{21})\right]^2 > 0, \qquad \text{with,} \tag{6b}$$

$$\text{in the non-integrable case:} \quad a_{12} \neq a_{21}, \qquad \text{and} \tag{6c}$$

$$\text{in the integrable case:} \quad a_{12} = a_{21}. \tag{6d}$$

Furthermore, the terms of higher order, $O(|\mathbf{y}|^2)$, must be such as to not violate conditions (A–B) for any $\mathbf{y} \in Y$.

Notice that we only speak here of a "candidate" gradient field. Indeed, in order for such a candidate to be effectively generated by a NTSP, it must have a proper "continuation" on all other budget constraints. It is conjectured, but not formally proven, that such a continuation will always exist.

Given this extreme latitude on the possible form of the field $\mathbf{q}$, it is impossible to address the problem of the global failure of local preference optimization in all its generality, at least from the "constructive" perspective followed in the present contribution. (General existence results for cycles in local preference optimization are given in [1] for the case of 4 or more goods. However, the topological approach followed in this reference cannot be applied to the three-goods case, does not give any insights as to the type of cycles arising, and does not permit to more precisely characterize the conditions necessary for cycles or to recognize whether cycles are common and robust occurences.) The best we can hope for is to show that there exists a dense class of gradient fields and search processes compatible with the NTSP theory under which local optimization globally fails. This is the approach we are following here. As previously mentioned, we want to conduct the analysis from the viewpoint of one of the few results about the global behavior of a planar flow, namely, the much celebrated Poincaré-Bendixson theorem; see, e.g., [17, p. 19]:

THEOREM (POINCARÉ-BENDIXSON). *A nonempty, compact limit set of a C^1 planar dynamical system, which contains no equilibrium point, is a closed orbit.*

An important corollary of the Poincaré-Bendixson theorem is that a non-empty, compact set which is positively or negatively invariant contains either a limit cycle or a fixed point. In particular, a non-empty, compact, positively invariant set surrounding an unstable equilibrium contains (at least) one limit cycle. If this cycle is unique, it is stable. However, there may exist an odd number of limit cycles which are alternatively stable and unstable. Also, one cannot exclude the possibility of regions of bounded orbits bounded by two semi-stable limit cycles. However, this last configuration is structurally unstable; see, e.g., [18, Chapter X].

With regard to the consumer's search problem, the Poincaré-Bendixson Theorem and its corollaries thus imply the following. If, for some search process $\frac{d\mathbf{y}_t}{dt}$, the equilibrium $\mathbf{y}^*$ is unique and repellent, and if there exists a compact region R including $\mathbf{y}^*$ such that $\frac{d\mathbf{y}_t}{dt}$ points towards R for all $\mathbf{y}$ not in R, then the search process will approach a limit cycle around the equilibrium as $t \to \infty$. This, independently from the point $\mathbf{y}^0$ from which the search process starts.

Before turning to the questions of the stability and uniqueness of the critical point of a positive search process, and of the existence of an appropriate region R, we now discuss how to construct positive search processes or, more precisely: a class of candidate positive search processes.

3.2. Positive Directions With Respect to the Linear Part

Even in the case of "simple" terms higher order, it is very difficult, if not impossible, to analytically determine whether a given non-gradient field $\mathbf{m}(\mathbf{y})$ is positive at every $\mathbf{y} \in Y$, with respect to the projected gradient field $\mathbf{q}(\mathbf{y})$ of a given preference in NTSP (from now on, we neglect the time subscript t in the notation). However, it is easy to fully characterize the class of all directions $\mathbf{m}_L(\mathbf{y})$ which are positive with respect to the *linear* part $\mathbf{A}\,\mathbf{y}$, $\mathbf{A}\,\mathbf{y} \neq \mathbf{0}$, of $\mathbf{q}(\mathbf{y})$. Indeed, any such direction $\mathbf{m}_L(\mathbf{y})$ can be expressed as:

$$\mathbf{m}_L(\mathbf{y}) = \mathbf{M}(\mu,\theta)\,\mathbf{y} \tag{7}$$

with:

$$\mathbf{M}(\mu,\theta) = \begin{bmatrix} \mu\, a_{11} - \theta\, a_{21} & \mu\, a_{12} - \theta\, a_{22} \\ \mu\, a_{21} + \theta\, a_{11} & \mu\, a_{22} + \theta\, a_{12} \end{bmatrix}, \qquad \mu \in \mathbb{R}_{++}, \quad \theta \in \mathbb{R}, \quad \theta \neq 0. \tag{8}$$

PROOF. Specialization to the case $n = 2$ of the corresponding results in [2, Section 9].

Vividly speaking, this reflects the fact that the set of all positive directions with respect to $\mathbf{A}\,\mathbf{y}$ is the interior of the one of the two halfspaces defined by the orthogonal complement of $\mathbf{A}\,\mathbf{y}$ (that is, by the line $\beta\,\mathbf{M}(0,\theta)\,\mathbf{y}$, $\beta \in \mathbb{R}$) in which $\mathbf{A}\,\mathbf{y}$ lies; see Figure 1. Thus, modulo a positive scalar multiple, any direction $\mathbf{m}_L(\mathbf{y})$ which is positive with respect to $\mathbf{A}\,\mathbf{y}$ can be expressed as a linear combination of $\mathbf{A}\,\mathbf{y}$, weighted by some strictly positive weight μ, and of a given element $\mathbf{M}(0,\theta)\,\mathbf{y}$ of $\mathbf{A}\,\mathbf{y}$'s orthogonal complement. For any given $\theta \neq 0$, the larger μ, the closer $\mathbf{m}_L(\mathbf{y})$ is to $\mathbf{A}\,\mathbf{y}$. When $|\mu/\theta| \to \infty$, then $\mathbf{m}_L(\mathbf{y})$ tends to point in the same direction as $\mathbf{A}\,\mathbf{y}$. When $|\mu/\theta| \to 0$, then $\mathbf{m}_L(\mathbf{y})$ tends to point along the orthogonal complement of $\mathbf{A}\,\mathbf{y}$.

Thus motivated, we assume that the consumer follows a search process of the type:

$$\frac{d\mathbf{y}}{dt} = \mathbf{m}(\mathbf{y}) = \mathbf{m}_L(\mathbf{y}) + O(|\mathbf{y}|^2), \tag{9}$$

with $\mathbf{m}_L(\mathbf{y})$ defined by assigning fixed values to μ and θ in (8).

Obviously, $\mathbf{y}^*$ is an equilibrium, but not necessarily the unique equilibrium, of the dynamic system defined by (9). More importantly, the thus-defined field $\mathbf{m}(\mathbf{y})$ is only a candidate positive process, since it does not need to be positive with respect to the projected gradient field $\mathbf{q}(\mathbf{y})$ at some or all $\mathbf{y} \in Y$ even if it is positive with respect to the linear part $\mathbf{A}\,\mathbf{y}$. That is, the scalar product $\mathbf{m}(\mathbf{y})\,\mathbf{q}(\mathbf{y}) = [\mathbf{m}_L(\mathbf{y}) + O(|\mathbf{y}|^2)]\,[\mathbf{A}\,\mathbf{y} + O(|\mathbf{y}|^2)]$ can be zero or negative even

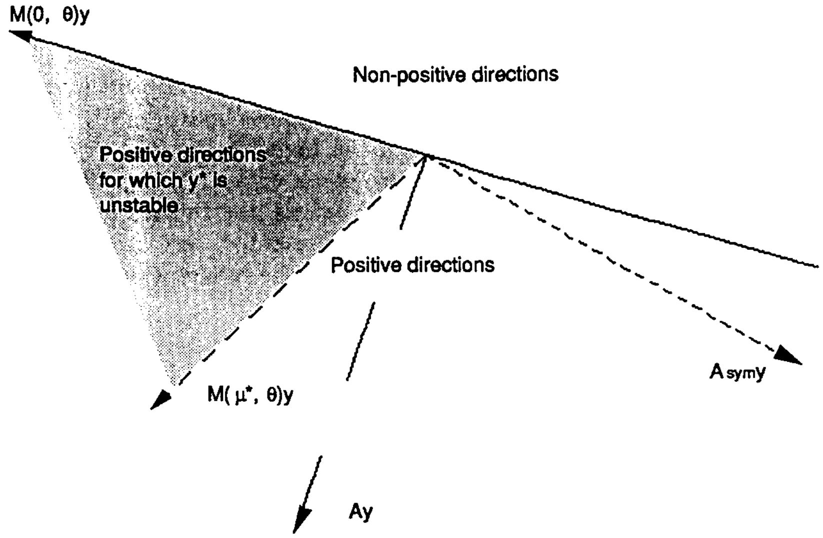


Figure 1. Relative position of the diverse "critical" directions.

if $\mathbf{M}(\mu,\theta)\,\mathbf{y}\,\mathbf{A}\,\mathbf{y}$ is positive. Unfortunately, as already mentioned, it appears impossible to give conditions on $\mathbf{m}_L(\mathbf{y})$ and/or $O(|\mathbf{y}|^2)$ which insure positivity of $\mathbf{m}(\mathbf{y})$ over the budget constraint Y. Indeed, we shall not try to derive such conditions, but will simply attempt to numerically show (Section 3.4) that there exist compact sets of matrices $\mathbf{M}(\mu,\theta)$ and of terms higher order than $O(|\mathbf{y}|^2)$, for which the uniqueness and positivity requirements are satisfied.

3.3. Stability of the Equilibrium

We now conduct a local stability analysis of the equilibrium $\mathbf{y}^* = \mathbf{0}$ in terms of the hypothesized search process (9). Above all, we want to give precise conditions under which $\mathbf{y}^*$ is a repellent. Not surprisingly, the results we shall obtain closely parallel (some of) the conditions for the emergence of a Hopf bifurcation at $\mathbf{y}^*$ derived in [2].

Generally speaking, $\mathbf{y}^*$ will be a repellent for the search process IFF (see, e.g., [19]):

$$\operatorname{tr}\mathbf{M}(\mu,\theta) > 0, \quad \text{that is,} \quad (a_{11}+a_{22})\,\mu + (a_{12}-a_{21})\,\theta > 0, \qquad \text{and} \tag{10a}$$

$$\det\mathbf{M}(\mu,\theta) > 0, \quad \text{that is,} \quad (\mu^2+\theta^2)\det\mathbf{A} > 0. \tag{10b}$$

Condition (6b) implies that (10b) is satisfied for all μ, $\theta \neq 0$. From (6a), one follows that (10a) and the requirement $\mu \in \mathbb{R}_{++}$ will be satisfied exactly if $a_{12} \neq a_{21}$, that is, if the preferences are not integrable, and if

$$\operatorname{sign}\theta = \operatorname{sign} a_{12} - a_{21}, \qquad \text{and} \tag{11a}$$

$$\mu = -\left[\frac{a_{12}-a_{21}}{a_{11}+a_{22}}\right]\theta - \alpha \equiv \mu^*\,\theta - \alpha, \quad 0 < \alpha < -\left[\frac{a_{12}-a_{21}}{a_{11}+a_{22}}\right]\theta \equiv \alpha^* \tag{11b}$$

Note that an increasing value of α implies a decreasing value of μ and of $|\mu/\theta|$, with the already outlined consequences for the direction of $\mathbf{m}_L$.

Condition (10a) cannot be satisfied when the preferences are integrable. Furthermore, for any θ satisfying (11a) and $0 \le \mu < \mu^*$, $\operatorname{tr}\mathbf{M}(\mu,\theta) < 0$ and $\det\mathbf{M}(\mu,\theta) > 0$, so that $\mathbf{y}^* = \mathbf{0}$ is a stable equilibrium. Thus, one needs *both* intransitivity *and* "inexactitude" for $\mathbf{y}^*$ to be repellent. An inexact but transitive consumer, as well as an exact but intransitive one, will be always attracted towards $\mathbf{y}^*$, at least locally. One recognizes that instability requires a higher degree of imprecision, as expressed by $|\theta/\mu|$, when $|a_{11}+a_{22}|$ is large compared to $|a_{12}-a_{21}|$. Thus, it appears natural to interpret $|\mu^*/\theta|$ as a measure of consumer intransitivity; see [2]. The corresponding "measure," however, becomes much more complicated in the case $n > 2$; see [3].

It is easy to recognize from the above that, geometrically speaking, a direction is a positive direction $\mathbf{m}_L(\mathbf{y})$ AND satisfies (10) IFF it lies between $\mathbf{M}(\mu^*,\theta)\,\mathbf{y}$ and $\mathbf{M}(0,\theta)\,\mathbf{y}$. Note that $\mathbf{M}(\mu^*,\theta)\,\mathbf{y}$, and thus any "unstable" positive direction $\mathbf{m}_L(\mathbf{y})$, will always lie on the other side of the line $\beta\,\mathbf{A}\,\mathbf{y}$, $\beta \in \mathbb{R}$, then $\mathbf{A}_{sym}\mathbf{y} \equiv \frac{1}{2}(\mathbf{A}+\mathbf{A}^\top)\mathbf{y}$, with $\mathbf{A}^\top$ denoting the transposed of A—see Figure 1.

3.4. Numerical Results

At this point, we are led to join Mira [20, p. 28] in recognizing that "Unfortunately ... a recourse to approximate methods both analytical and numerical becomes unavoidable." In particular, even under the simplest hypotheses concerning the terms of higher order, the expressions $\mathbf{m}(\mathbf{y})\,\mathbf{q}(\mathbf{y})$ are complicated polynomials whose coefficients cannot be univocally signed on the basis of the diverse assumptions presented earlier. Thus, *volens nolens*, the further insight on the possible global failure of local preferences optimization to be presented here stems from a still rather restricted numerical study, which was carried out under the assumption of higher order terms $O(|\mathbf{y}|^2)$ of the form:

$$O(|\mathbf{y}|^2) = -\gamma \begin{bmatrix} y_1^3 \\ y_2^3 \end{bmatrix}, \qquad \gamma \in \mathbb{R}_{++}. \tag{12}$$

Besides approximating large classes of other possible higher order terms, the formulation (12) insures that the Jacobian of $\mathbf{A}\,\mathbf{y} + O(|\mathbf{y}|^2)$ satisfies the conditions (A–B) for any $\mathbf{y}$, if $\mathbf{A}$ satisfies these conditions. Moreover, since the square of the higher order terms enters additively the scalar product $\mathbf{m}(\mathbf{y})\,\mathbf{q}(\mathbf{y})$ for any search process $\mathbf{m}(\mathbf{y})$ satisfying (9), the hypothesis (12) guarantees that this search process will be positive for $\mathbf{y}$ sufficiently large. Similarly, equation (12) insures that for $\mathbf{y}$ sufficiently large, the $-\gamma\, y_i^3$ terms dominate the corresponding coordinates of $\mathbf{m}(\mathbf{y})$. That is, it insures that the vectors $\mathbf{m}(\mathbf{y})$ will all point towards a compact region R containing $\mathbf{y}^*$. Thus, by the theorem of Poincaré-Bendixson, any search process originating at $\mathbf{y}^0 \neq \mathbf{y}^*$ will approach a closed trajectory within R if this region does not contain any equilibrium point besides the unstable $\mathbf{y}^*$. Therefore, for any matrix $\mathbf{A}$, equation (12) allows us to restrict the numerical investigations to a domain R' such that $R \subset R'$. In almost all the cases studied, it proved sufficient to choose as "default" for R' the square defined by $y_i \in [-3, 3]$, $i = 1, 2$. (The adequacy of this choice was checked in each case by numerically verifying that the 0.1 level curve $\{y \mid \mathbf{m}(\mathbf{y})\,\mathbf{m}(\mathbf{y}) = 0.1\}$, for the Lyapunov-type function $\mathbf{m}(\mathbf{y})\,\mathbf{m}(\mathbf{y})$, lay completely within R'. It should be clear that the choice of this particular definition of R', and the use of R' instead of R, has no substantive signification. It was done solely in order to keep, without loss of generality, the numerical investigation simple.)

All computations were done on a Macintosh SE/30 using MATHEMATICA version 1.2. Due to an apparent bug in this version, the study was limited to matrices $\mathbf{A}$ such that $\operatorname{sign} a_{12} = \operatorname{sign} a_{21}$. Also (and for obvious substantive reasons), only matrices $\mathbf{A}$ satisfying condition (A) and values of α and γ such that $0 < \alpha < \alpha^*$, $\gamma > 0$ were taken into consideration. Without loss of generality, θ was always given the value $\operatorname{sign} a_{12} - a_{21}$.

Motivated by the previous discourse, the fundamental question for us was:

Are there dense subsets of the triples $(\mathbf{A}, \alpha, \gamma)$ such that the corresponding search processes $\mathbf{m}(\mathbf{y})$ are globally positive (that is, positive for all $\mathbf{y} \in Y$) and do not have other singular points than $\mathbf{y}^*$?

In our attempts to answer this question, we started with two triples which generate globally positive search processes with the unique singular point $\mathbf{y}^* = 0$, namely:

$$\mathbf{A} = \pm \begin{bmatrix} -4 & 1 \\ 3 & -4 \end{bmatrix}, \qquad \alpha = 0, \quad \gamma = 1.$$

These triples were then systematically modified by letting:

(a) a_{11} and a_{22} increase/decrease in steps of $\pm 50\%$, and a_{12} (respectively, a_{21}) in steps of $\pm 20\%$ from their current values, over 20 steps in each direction or until (A) was violated, whichever came first.

All possible combinations of the modifications described under (a) were generated (that is, for example, the combination with a_{11} and a_{22} kept at its base value, a_{12} decreased by 20%, a_{21} increased by 44%, etc.). Furthermore, for each of the thus generated matrices $\mathbf{A}$:

(b) γ was given the values 0.1, 10, 100, 1000, 100000;
(c) α was increased from 0 to the corresponding α^* by steps of $\alpha^*/10$.

The positivity of the search processes $\mathbf{m}(\mathbf{y})$ and the uniqueness of the singular point was checked in each case—the former with the help of the MATHEMATICA `DensityLevel` function with `PlotPoints -> 100`, the later by computing the solutions of $\frac{d\mathbf{y}}{dt} = 0$ with `Solve//N` and verifying that they did not differ coordinatewise from 0 by more than 1×10^{-15}. Furthermore, 2×2 matrices were generated randomly, with elements uniformly independently distributed over $[-100, 100]$, and with diagonal elements a_{ii} (which, in the NTPS case, are necessarily negative) uniformly independently distributed over $[-100, 0]$. The positivity/uniqueness check was carried out on the first 200 of these matrices which satisfied (A) and the sign restriction $\operatorname{sign} a_{12} = \operatorname{sign} a_{21}$, for the values of γ and α indicated under (b) and (c).

This exercise suggests the following, still tentative general qualitative hypotheses:

(i) When the coefficients of $\mathbf{A}$ are relatively "balanced," in the sense that a_{11} does not differ too much from a_{22}, and a_{12} from a_{21}, and when $|a_{11} + a_{22}|$ is large compared to $|a_{12} - a_{21}|$ (that is, the consumer intransitivity is low), there always exists a large interval $[0, \alpha_{\max} < \alpha^*]$ of α-values for which global positivity and equilibrium uniqueness are given. This interval is the larger, the lower the consumer intransitivity is. In "well-balanced, low intransitivity" cases, $\alpha_{\max}$ is arbitrarily close to α^*.

Thus, it appears that our fundamental question can be conclusively answered. Furthermore:

(ii) For all $\alpha > \alpha_{\max}'$, $\alpha_{\max}' \geq \alpha_{\max}$, one typically observes the emergence of other singular points than $\mathbf{y}^*$, namely, generically, either (iia) two stable or (iib) two stable and two unstable equilibria. However, for these values of α, the search processes are no longer globally positive.

Thus, the emergence of a stable singular point besides the unstable $\mathbf{y}^*$ appears incompatible with the hypothesized consumer behavior, at least in the particular specification (9)/(12). Note that we observed global positivity failures even with $\mathbf{y}^*$ as the unique singular point, see (iii), but never global positivity under multiple equilibria. Also:

(iii) For any value of α and γ, global positivity appears incompatible with a markedly unbalanced and/or intransitive matrix $\mathbf{A}$.

(iv) Increasing values of γ contribute towards global positivity in the sense that, for given values of the a_{ij}'s, $\alpha_{\max}$ is increasing in γ. Similarly, a greater imbalance of $\mathbf{A}$ is compatible with global positivity/uniqueness.

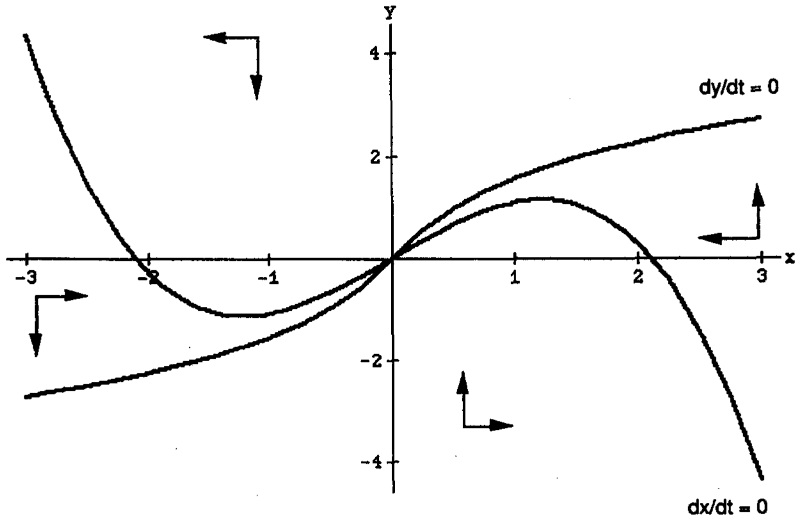
This last result reflects the fact that, as γ increases, the relative importance of the linear part decreases for any value of $\mathbf{y}$, so that the search process and gradient field tend to coincide everywhere.

No attempt was made to go behind these general qualitative statements to, for example, derive frontiers between those parameters constellations for which global positivity/uniqueness is given and those for which it is not. Similarly, we did not endeavor to economically interpret the results—something which anyway might well prove more contrived than truly enlightening.

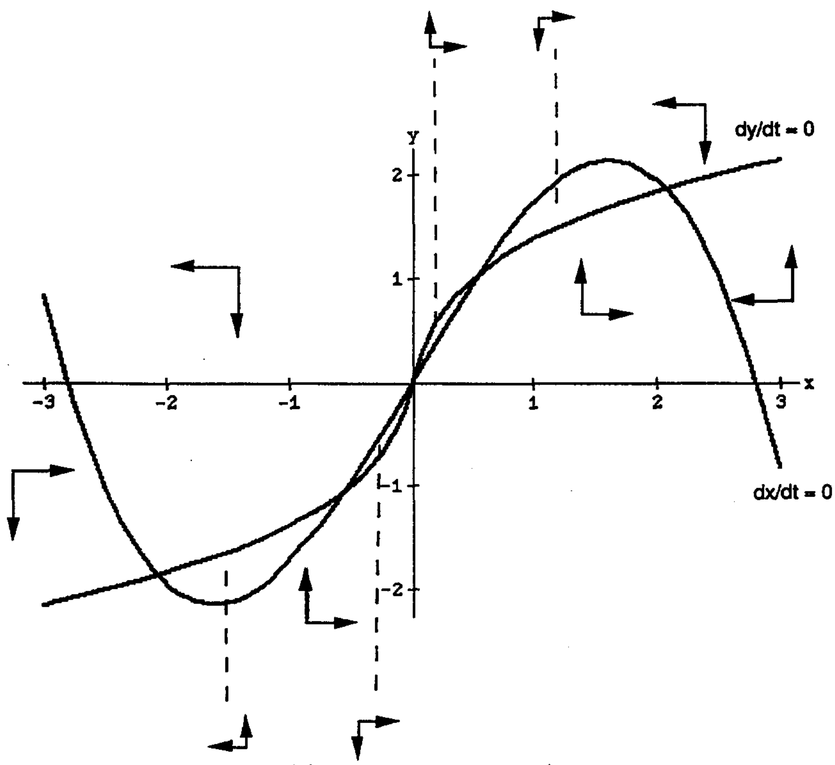
Figure 2 shows the phase plane for the search process $\mathbf{m}(\mathbf{y})$ in the case $\mathbf{A} = \begin{bmatrix} -4 & 1 \\ 6.9 & -4 \end{bmatrix}$, $\gamma = 1$, and $\alpha = 0$ (Figure 2a) or $\alpha = 0.99\alpha^*$ (Figure 2b). Notice that in this example $\mathbf{A}$ is strongly unbalanced. Indeed, it would violate condition (A) if either a_{12} or a_{21} were increased by 0.1. Also, be aware of the fact that on the figure, as in the following commentary, x is used to designate the first and y the second coordinate of the consumption bundle $\mathbf{y}$.

The two phase diagrams show the results mentioned under (ii), specifically, under (iib). For $\alpha = 0$, the origin is the only singular point. Moreover, the search process is globally positive. For $\alpha \cong 0.6\,\alpha^*$, there is a non-generic, borderline case (not shown) when the $\frac{dx}{dt} = 0$ locus becomes tangent at two points with the $\frac{dy}{dt} = 0$ locus, defining two new, stable critical points. For larger values of α, the $\frac{dx}{dt} = 0$ locus intersects the $\frac{dy}{dt} = 0$ locus in 5 points. The two points closer to the origin are unstable, the other two stable. However, for $\alpha > 0.6\,\alpha^*$, the process is no longer globally stable. This is illustrated for $\alpha = 0.99\,\alpha^*$ in Figure 3, where the region where $\mathbf{m}(\mathbf{y})\,\mathbf{q}(\mathbf{y}) < 0$ is shown in black.

According to our investigations, it is likely that the shape of the $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ loci in Figure 2 is generic for the class of systems studied; changing parameter values, the slope and shape of the



(a) Phase diagram, $\alpha = 0$.



(b) Phase diagram, $\alpha \cong \alpha^*$.

Figure 2.

loci smoothly vary. Replacing the given a_{ij}, $i \neq j$, by $-a_{ij}$ leads to a phase plane which mirrors the one shown here, and so on. Modulo such rather trivial modifications, however, we found qualitatively similar phase portraits in all the cases (about 50) for which they were explicitly drawn.

In any event, it should be clear that the only evidence we have on the shape of the isoclines stems from the rather limited numerical investigations we presented. Indeed, the definitory equations for $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$:

$$y = -\frac{x\,(\alpha\, a_{11}^2 + \sigma\, a_{11}\, a_{12} + \alpha\, a_{11}\, a_{22} + \sigma\, a_{21}\, a_{22} + \gamma\, a_{11}\, x^2 + \gamma\, a_{22}\, x^2)}{(\alpha\, a_{11}\, a_{12} + \sigma\, a_{12}^2 - \sigma\, a_{12}\, a_{21} + \sigma\, a_{11}\, a_{22} + \alpha\, a_{12}\, a_{22} + \sigma\, a_{22}^2)}, \qquad \text{and} \qquad (13)$$

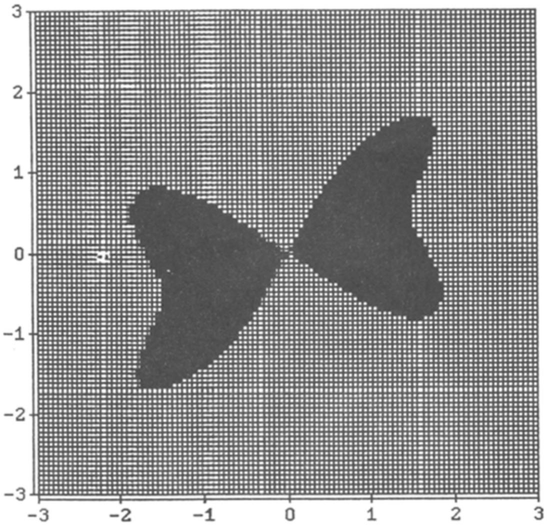


Figure 3. Zones where positivity is violated, $\alpha \cong \alpha^*$.

$$x = \frac{y\,(-\sigma\, a_{11}\, a_{12} + \alpha\, a_{11}\, a_{22} - \sigma\, a_{21}\, a_{22} + \alpha\, a_{22}^2 + \gamma\, a_{11}\, y^2 + \gamma\, a_{22}\, y^2)}{(\sigma\, a_{11}^2 - \alpha\, a_{11}\, a_{21} - \sigma\, a_{12}\, a_{21} + \sigma\, a_{21}^2 + \sigma\, a_{11}\, a_{22} - \alpha\, a_{21}\, a_{22})}, \tag{14}$$

where

$$\sigma \equiv \operatorname{sign} a_{12} - a_{21}, \tag{15}$$

are such that we were unable to analytically delineate the absolute and relative isocline behavior possible under the problem-endogenous restrictions on α, γ, and the a_{ij}'s.

4. CONCLUSIONS

The analyses presented in this article suggest that (on the current budget set) an intransitive inexact consumer following a fairly general class of positive search process will globally fail to find his consumption optimum for a compact range of the relevant behavioral parameters. This phenomenon will always occur (and be most robust) in the mundane cases when the consumer's preferences are neither strongly intransitive nor drastically unbalanced, and his inexactitude is sufficiently but not extremly high.

Other specifications of the positive search process may, of course, prove compatible or incompatible with the robust global failure of the local preferences optimization. Exhaustively exploring specific functional forms and parameter ranges under which the one or the other is true, along lines similar to those followed in this article, is not only a hopelessly unbounded task, it is also futile, to the extent that we do not have pertinent theoretical or empirical guidance as to the kind of search processes, if any, consumers do follow in reality.

Thus, the most that this article can hope to achieve is to once again bring to the forefront the fact that not only do "optimal equilibria" matter, but also the process by which they are, or are not, reached. Moreover, if its arguments are taken seriously, the article casts a reasonable doubt about the concept of consumer optimum as routinely used in consumption theory. (This precise point, interestingly enough, was taken by a referee to one of the author's previous articles on the subject as the main argument for the rejection he unsuccessfuly advocated.)

Assuredly, the arguments presented here are incomplete in at least one respect. It is unreasonable to imagine a consumer who, as implicitly suggested by the previous argumentation, would engage into an endless, cyclical search process. In real life, consumption decisions are always taken, in finite time. Thus, the search processes defined in this article are certainly oversimplified, and need to be refined within the framework of a general theory of search and decision-making for intransitive consumers. Without speculating on the possible outcomes of such a study, one cannot exclude that it would reconfirm the supremacy of the classical consumer optimum by suggesting search mechanisms which, thanks to learning phenomena for example, always lead to the unique global optimum. However, it may also support the hypothesis that, even in non-pathological situations, real consumers often make choices far away from the theoretical consumption optimum—for example, whenever the time needed to find the ideally best consumption bundle is large compared to the consumer's planning horizon. Such a behavior, obviously, introduces a fundamental indeterminacy in consumer choice theory—with a subsequent need for re-examining both the theoretical role and the practical value of the classical consumer optimum in economics.

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